What is Renormalization?

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A History (of Sorts)

An example:

$$\frac{g_e}{2} = 1 + \frac{\alpha_0}{2\pi} + \infty\alpha_0^2 + \infty^2\alpha_0^3 + \cdots$$

but

$$\alpha_0 = \frac{\alpha}{1 - \infty\alpha - \infty^2\alpha^2 - \cdots}$$

$$= \alpha\,(1 + \infty\alpha + \cdots)$$

Taylor expansion in powers of $\infty\alpha$.

implies

$$\frac{g_e}{2} = 1.000579826087\ldots$$

(experiment $\Rightarrow$ 1.000579826087...)

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$$\times \int \frac{d^d q'}{(2\pi)^d} e^{-yq'^2} \left[-4(d-2)A_1 + \cdots - \frac{4(d-2)}{4\sum^2} C_2 \right] \tag{8-124}$$

It is then a pure matter of patience to compute

$$\begin{aligned}
(4\pi y)^{d/2} \int \frac{d^d q'}{(2\pi)^d} e^{-yq'^2} A_1 = {} & k_\rho k_\sigma \frac{\alpha_{23}\alpha_{14} + 2(\alpha_1 - \alpha_3)(\alpha_4 - \alpha_2)}{2y\sum^2} (d-2) \\
& + \delta_{\rho\sigma} \Bigg[(k^2)^2 z(1-z)(\alpha+z)(1-\alpha-z) \\
& + \frac{k^2}{2y} \left\{ 2\hat{\alpha}_{23}(1-\hat{\alpha}_{23})[\alpha + 2z - 2z(\alpha+z)] + \frac{2(\alpha_1-\alpha_3)(\alpha_4-\alpha_2)}{\sum^2} \right\} \\
& - \frac{k^2 d}{2y} \left[(\alpha+z)(1-\alpha-z)\hat{\alpha}_{23}^2 + (1-\hat{\alpha}_{23})^2 z(1-z) \right] \\
& + \frac{d(d+2)}{4y^2} \hat{\alpha}_{23}^2 (1-\hat{\alpha}_{23})^2 \Bigg]
\end{aligned}$$

$$\begin{aligned}
(4\pi y)^{d/2} \int \frac{d^d q'}{(2\pi)^d} e^{-yq'^2} B_1 = {} & (2-d)(2\alpha^2 - 1) k_\rho k_\sigma \\
& - (d-2)\delta_{\rho\sigma} \left\{ \frac{1}{y} + k^2[(\alpha+z)(1-\alpha-z) + z(1-z)] - \frac{d}{2y}\left[\hat{\alpha}_{23}^2 + (1-\hat{\alpha}_{23})^2\right] \right\} \\
& + 2\delta_{\rho\sigma} \left[k^2(1-2z)(1-2z-2\alpha) - 2\hat{\alpha}_{23}(1-\hat{\alpha}_{23}) \frac{d}{y} \right]
\end{aligned}$$

$$(4\pi y)^{d/2} \int \frac{d^d q'}{(2\pi)^d} e^{-yq'^2} A_2 = \frac{d-2}{2y} \hat{\alpha}_{23}(1-\hat{\alpha}_{23})(k_\rho k_\sigma - k^2 \delta_{\rho\sigma})$$

Consequently,

$$\begin{aligned}
\Gamma_{\rho\sigma}^{(A_2+B_2)} = {} & -\frac{e^4}{(4\pi)^d} 4(6-d)(d-2)(k^2\delta_{\rho\sigma} - k_\rho k_\sigma) \int_0^1 \frac{dx'}{x'^2} \int_0^\infty \frac{d\alpha_1 \cdots d\alpha_4 \alpha_{23}\alpha_{14}(1-x')}{2\sum^{d/2+2} y^{d/2}} \\
& \times \exp\left[-\frac{\alpha_{12}\alpha_{23}\alpha_{34}\alpha_{41} - x'(\alpha_1\alpha_3 - \alpha_2\alpha_4)^2}{\sum \alpha_{23}\alpha_{14}} k^2 \right] \\
= {} & -\frac{e^4}{(4\pi)^d} 2(6-d)(d-2)(k^2\delta_{\rho\sigma} - k_\rho k_\sigma)(k^2)^{d-4}\Gamma(4-d) \\
& \times \int_0^1 dx'\, x'^{d/2-2}(1-x') \int_0^1 \frac{d\alpha_1 \cdots d\alpha_4 \delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)}{(\alpha_{14}\alpha_{23})^{3d/2-5}} \\
& \times \left[\alpha_{12}\alpha_{23}\alpha_{34}\alpha_{41} - x'(\alpha_1\alpha_3 - \alpha_2\alpha_4)^2\right]^{d/2-4}
\end{aligned}$$

After the change of variables,

$$\alpha_1 = \beta u \qquad \alpha_2 = (1-\beta)v \qquad \alpha_3 = (1-\beta)(1-v) \qquad \alpha_4 = \beta(1-u)$$

$$\alpha_{14} = \beta \qquad \alpha_{23} = 1 - \beta$$

$$\begin{aligned}
[\alpha_{12}\alpha_{23}\alpha_{34}\alpha_{41} - x'(\alpha_1\alpha_3 - \alpha_2\alpha_4)^2] = {} & \beta(1-\beta)\{[\beta(1-u) + (1-\beta)(1-v)] \\
& \times [\beta u + (1-\beta)v] - x'\beta(1-\beta)(u-v)^2\}
\end{aligned}$$

$$\int_0^1 d\alpha_1 \cdots d\alpha_4 \delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4) F(\alpha_i) = \int_0^1 d\beta\, \beta(1-\beta) \int_0^1 du \int_0^1 dv\, F(\alpha_i)$$

Q. Why is QED renormalizable?

A1. Only thing we can make sense of?

A2. New axiom of nature:

"All physical field theories are renormalizable!?"

$\Rightarrow$ $SU_2 \times U_1$ weak interactions
$\Rightarrow$ SU_3 strong interactions
$\Rightarrow$ ...

A3. Or...

- Axiom is unnecessary!
- Probably no current theory that is exactly renormalizable!

The Idea

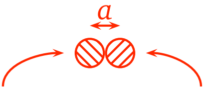
Quantum Electrodynamics

$\mathbf{E}(\mathbf{x}, t)$ = quantum mechanical operator

$\Rightarrow$ Measurements of **E** fluctuate from measurement to measurment.

$\Rightarrow$ $\mathbf{E}(\mathbf{x}, t)$ fluctuates from point to point.

Eg) Electric field averaged over probe size a:



$\bar{E}(x)$ $\bar{E}(x+a)$

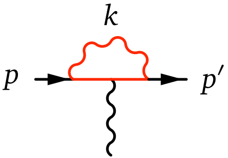
$$\left\langle \left(\bar{E}(x) - \bar{E}(x+a)\right)^2 \right\rangle \to \frac{1}{a^4} \quad \text{as } a \to 0$$

$\Rightarrow$ $\mathbf{E}(\mathbf{x}, t)$ is infinitely rough at short distances! Derivatives??

$\Rightarrow$ Quantum field theories have structure at arbitrarily short distances! Problem?

Does it matter?

Eg)



$$= \int d^4k \ldots$$

⇒ Integral diverges from $k \to \infty$ states.

⇒ $k \to \infty$ states infinitely important?

⇒ Need to understand string/M theory (or. . . ?) in order to calculate anything?? Disaster???

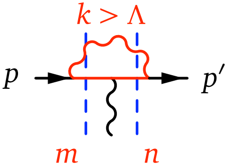
UV Cutoff

- Introduce UV cutoff: omit all states with $k > \Lambda$ from theory.
- Choose $\Lambda \gg p$ where

$$p = \text{typical momentum in process of interest,}$$

  but $\Lambda \not\to \infty$!!
- Fixes infinities, but ...

What is left out?

Eg) 

$k > \Lambda \gg p, p' \Rightarrow$ states m, n far off shell ($\Delta E \approx \Lambda$).

$\Rightarrow$ m, n very shortlived (uncertainty principle):

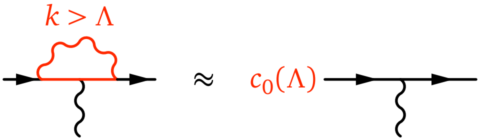
$$\Delta t \approx \frac{1}{\Delta E} \approx \frac{1}{\Lambda}.$$

$\Rightarrow$ Interaction occurs over very small region:

$$\Delta x \approx \frac{1}{\Lambda} \ll \frac{1}{p}.$$

$\Rightarrow$ Interactions effectively local compared to $\lambda \approx 1/p$.

⇒ Can mimic piece of theory excluded by cutoff with new local interaction:



⇒ Add $k > \Lambda$ physics back in by adding

$$\delta\mathscr{L} \equiv c_0(\Lambda)\,\overline{\psi}\not{A}\psi$$

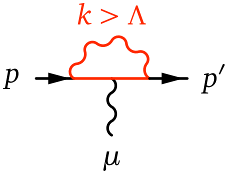
to the cutoff Lagrangian (much simpler)!

N.B. $\mathscr{L}^{(\Lambda)} + \delta\mathscr{L}$ then has interaction $e(\Lambda)\,\overline{\psi}\not{A}\psi$ where

$$e(\Lambda) \equiv e_0 + c_0(\Lambda) = \text{“running coupling.”}$$

More Accuracy

Taylor expand in p/Λ, p'/Λ:



$$= c_0(\Lambda)\ \bar{u}\gamma_\mu u$$

$$+ \frac{c_1(\Lambda)}{\Lambda}\ \bar{u}\sigma_{\mu\nu}(p-p')^\nu u$$

$$+ \frac{c_2(\Lambda)}{\Lambda^2}(p-p')^2\ \bar{u}\gamma_\mu u$$

$$+ \ldots$$

$\Rightarrow$ Add more corrections to $\mathscr{L}^{(\Lambda)}$:

$$\frac{c_1(\Lambda)}{\Lambda} \quad \overline{\psi}\sigma_{\mu\nu}F^{\mu\nu}\psi \qquad \text{for } p/\Lambda$$

$$\frac{c_2(\Lambda)}{2\Lambda^2} \quad \overline{\psi}i\partial_\mu F^{\mu\nu}\psi \qquad \text{for } (p/\Lambda)^2$$

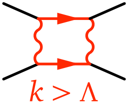
$$\vdots$$

N.B.

- Operators all local $\equiv$ polynomial in ψ, A_μ, and ∂_μ (Taylor expansion!).
- Infinitely many operators but only need first few since

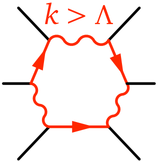
$$\frac{p}{\Lambda} \ll 1.$$

Only other amplitude important in order $1/\Lambda^2$ is



$$\rightarrow \quad \frac{d(\Lambda)}{\Lambda^2}(\overline{\psi}\gamma\psi)^2 + \cdots$$

Eg)



$$\rightarrow \quad \frac{f(\Lambda)}{\Lambda^5}(\overline{\psi}\gamma\psi)^3 + \cdots$$

$$\int_\Lambda d^4k \left(\frac{1}{\not{k}}\right)^3 \left(\frac{1}{k^2}\right)^2 \sim \frac{1}{\Lambda^5}$$

Note:

- Short-distance physics has a strong impact on long-distance physics. (c.f., UV divergences.)
- All we need to know about short distances is summarized in a finite number (determined by desired accuracy) of couplings — $c_1(\Lambda)$, $c_2(\Lambda)$, $e(\Lambda)$, $m(\Lambda)\ldots$ — for the cutoff theory. (c.f., multipole expansion.)
- Corrections non-renormalizable, but no infinities because cutoff $\Lambda \not\to \infty$.

- Form of $\delta\mathscr{L}$'s is independent of the dynamics for $k > \Lambda$! Only $c_1(\Lambda), c_2(\Lambda)\ldots$ care about details at $k > \Lambda$.

$\Rightarrow$ Don't need to understand gravity, string/M theory ...; the couplings parameterize our ignorance, and can be measured experimentally.

Summary: Renormalization Theory

- UV cutoff $\Rightarrow$ omit $k > \Lambda$ states
  $\Rightarrow$ no infinities
  $\Rightarrow$ no string/M theory needed!

- Add local universal correction terms, with theory-specific couplings, to $\mathscr{L}^{(\Lambda)}$ to mimic effects of $k > \Lambda$ physics.

- Only a finite number of correction terms needed for given accuracy, $(p/\Lambda)^n$.

$\Rightarrow$ Arbitrary precision with finite Λ!

Applications and Illustrations

Why is QED renormalizable?

QED = low-energy approximation to complex super-theory
(strings? branes? SUSY?) with threshold Λ.

$\Rightarrow$

$$\mathscr{L}_{\text{QED}}^{(\Lambda)} = \mathscr{L}_R + \frac{c_1\,\overline{\psi}\sigma\cdot F\psi}{\Lambda} + \frac{c_2\,\overline{\psi}\partial\cdot F\cdot\gamma\psi}{\Lambda^2} + \ldots$$

"Renormalizable" theory.

Λ is boundary between old and new physics.

Cutoff restricts theory to region of validity.

Due to new dynamics at $k > \Lambda$.

Terms really there, but only affect results in order $p/\Lambda \ll 1$.

$\Rightarrow$ Theory *appears* to be renormalizable!

Theorem

Very low-energy approximations to arbitrary high-energy dynamics can be described by renormalizable theories.

How renormalizable is QED?

Look for $1/\Lambda$ terms by

1. $p \approx \Lambda$ experiments $\Rightarrow$ $\mathcal{O}(1)$ effects but high cost (LHC).
2. $p \approx m$ experiments $\Rightarrow$ $\mathcal{O}(1)$ cost but tiny effects $(g_e - 2)$.

Eg) QED $\Rightarrow$ electron's mag. moment to $\delta\mu/\mu \approx 4 \times 10^{-12}$

- $(c/\Lambda)\,\overline{\psi}\sigma \cdot F\psi$ with c of $\mathcal{O}(1)$ $\Rightarrow$ $\delta\mu/\mu \approx m/\Lambda$
  $\Rightarrow$ $\Lambda > 10^8$ GeV!
- But chiral symmetry $\Rightarrow$ $c = \mathcal{O}(m/\Lambda)$
  $\Rightarrow$ $\Lambda > 10^3$ GeV.

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Proton QED?

$$\text{Experiment} \quad \Rightarrow \quad \delta\mu/\mu = \mathcal{O}(1)$$

$$\Rightarrow \quad m_p/\Lambda = \mathcal{O}(1)$$

$$\Rightarrow \quad \Lambda = \mathcal{O}(m_p) \text{ for new physics.}$$

Two consequences:

- Proton QED useless for $p = \mathcal{O}(m_p)$ since $\delta\mathscr{L}$s of all orders in p/Λ important. (Need QCD!)
- For $p \ll m_p$ (eg, atoms) proton QED can be made arbitrarily accurate by adding $\delta\mathscr{L}$s. (Don't need/want QCD!)

Atomic QED?

For H, Ps . . .

$$\text{Prob.}(p_e > m_e) \sim \alpha^5$$

- $\Rightarrow$ Atoms very non-relativistic.
- $\Rightarrow$ Choosing $\Lambda \approx m_e$ okay.
- $\Rightarrow$ Can use non-relativistic dynamics since $p_e > m_e$ states omitted.

QED → NRQED

$$
\begin{aligned}
\mathcal{L}^{(\Lambda)}_{\text{NRQED}} = \psi^\dagger \Bigg\{ & i\partial_t - e\phi + \frac{\mathbf{D}^2}{2m} \\
& - c_1 \frac{e}{2m} \sigma \cdot \mathbf{B} \\
& - c_2 \frac{e}{8m^2} \nabla \cdot \mathbf{E} \\
& + c_3 \frac{ie}{8m^2} \{\mathbf{E}\times, \mathbf{D}\} \cdot \sigma \\
& - \frac{\mathbf{D}^4}{8m^3} \Bigg\} \psi + \frac{d}{m^2} \psi^\dagger \sigma \psi \cdot \psi^\dagger \sigma \psi + \cdots
\end{aligned}
$$

Schrödinger Theory.

Relativistic corrections to Schrödinger Theory.
↓

e, m, c_1, c_2 ... chosen correctly ⇒ $\mathcal{L}_{\text{NRQED}} \equiv \mathcal{L}_{\text{QED}}$ for $p \ll m$.

Origin of $W^\pm/Z^0$ mass?

- Fermi Theory $=$ (contact interaction).
  - $\Rightarrow$ Non-renormalizable with $\Lambda \approx 100\,\text{GeV}$.
  - $\Rightarrow$ New physics $\approx 100\,\text{Gev}$: $W^\pm$ and Z^0.

- Minimal theory of $W^\pm/Z^0$ $=$ (Yang Mills $+$ mass term).
  - $\Rightarrow$ Non-renormalizable with

$$\Lambda \approx \frac{M_Z}{\sqrt{\alpha}} \approx 1\,\text{TeV}$$

  - $\Rightarrow$ Must see new physics by $\approx$ few TeV ($\Rightarrow$ LHC).

Light Higgs?

- Theory with light Higgs particle ($m \ll 1$ TeV) renormalizable but . . .
- unnatural unless cut off at $\Lambda \approx 1$ TeV.

$\Rightarrow$ New physics anyway!

Masses

Scale of couplings in $\mathscr{L}^{(\Lambda)}$ is set by Λ (or higher).

$\Rightarrow$ Bare masses (in lagrangian) $= \mathscr{O}(\Lambda)$.

$\Rightarrow$ Physical masses $= \mathscr{O}(\Lambda)$ barring miraculous (ie, unnatural) cancellation.

Theorem

If a particle has $m < 10^{19}$ GeV, there has to be a reason (symmetry): eg,

$$\begin{array}{rcl} \text{gauge symmetry} & \rightarrow & \text{spin } 1 \\ \text{chiral symmetry} & \rightarrow & \text{spin } 1/2 \\ & \vdots & \end{array}$$

Conclusion

- Renormalizability is not miraculous — approximate renormalizability a consequence of low-energy approximation.
- Important question is *not* “Is this theory renormalizable?” but rather “To what extent is this theory renormalizable?”.
  - More renorm’ble ⇒ larger range of validity (usually).
  - Corrections = model-indep. parameterization of new physics.
- For fundamental theories, “naturalness” is more important ⇒ symmetries are central.
- Theorists don’t have to apologize for renormalization any more; it is a powerful tool!